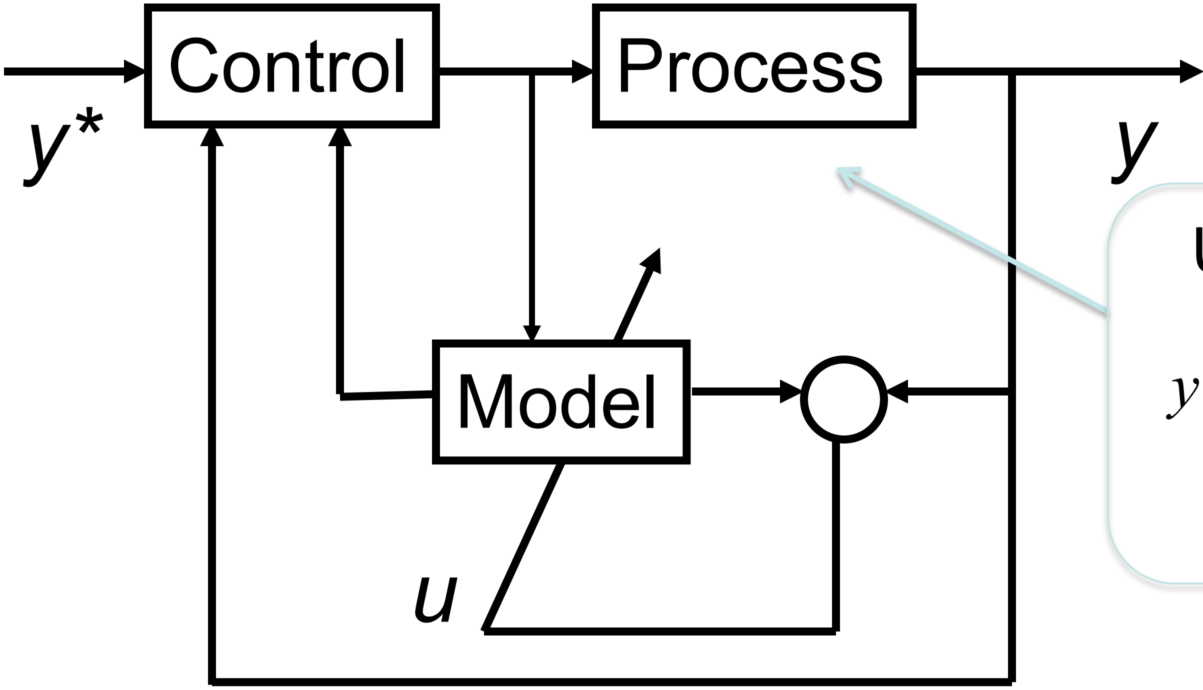
3.1 Adaptive Control Theory
Application in The Ideal Case

Objective: Review stability theory for the "ideal case" (no unknown disturbances or un-modelled dynamics)

1. Problem Statement
2. Realization Theory
3. Recursive Estimation
   a) Gradient Algorithm
   b) Recursive Least Squares
4. Direct Adaptive Control
   a) One step ahead control
   b) Adaptive Extended Horizon Control
5. Indirect Adaptive Control
   a) Adaptive EHC and Adaptive MPC
   b) Pole assignment

CE --Problem Statement



Unknown

$$y(t) = \frac{B(q^{-1})}{A(q^{-1})} u(t-d)$$

Problem: Design an Adaptive Controller which yields stable closed loop and converges to "optimal performance"

$$u(t) = \frac{1}{R(q^{-1})}\left(T(q^{-1})y^{*} - S(q^{-1})y(t)\right)$$

- Direct Adaptive Control – Estimate control parameters
- Indirect Adaptive Control – Estimate model and calculate controlle

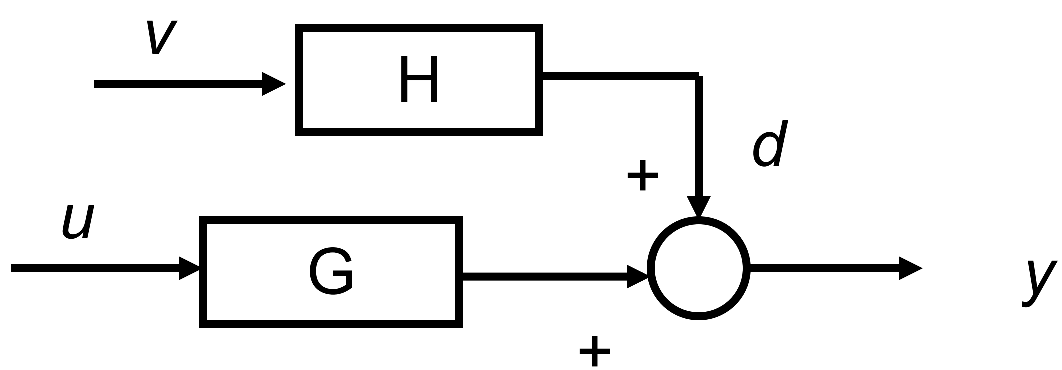
Transfer Functions:
Realization Theory

Notation:

$$qu(t) = u(t+1)$$

$$q^{-1}u(t) = u(t-1)$$

$$y(t) = \sum_{k=1}^{\infty} g(k)u(t-k) = \left(\sum_{k=1}^{\infty} g(k)q^{-k}\right)u(t) = G(q^{-1})u(t)$$

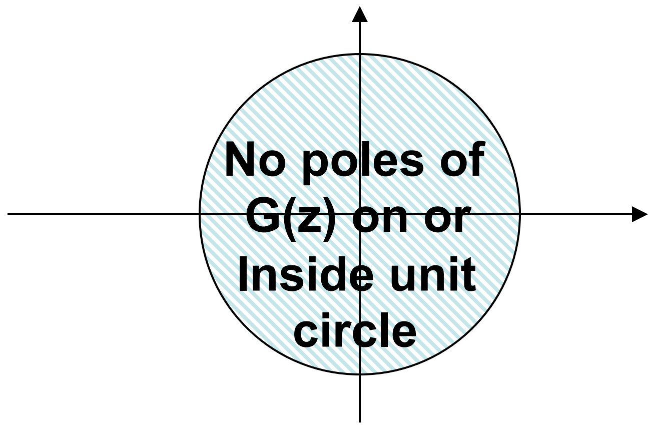


$$y(t) = G(q^{-1})u(t) + H(q^{-1})v(t)$$

Stability, Poles and Zeros

Stable (BIBO): $$\sum_{k=1}^{\infty} |g(k)| < \infty$$

Laurent expansion $G(z) = \sum g(k) z^{-k}$ converges for complex z



Solutions to $G(z) = 0$ are called the zeros

Solutions to $G(z) = \infty$ are called the poles

$G(z)$ stably invertible if $G(z)^{-1}$ is stable

Realization Theory

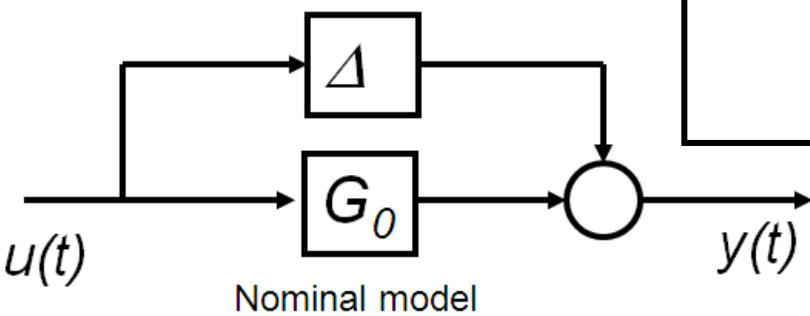
Question: How do we represent G and H?

Many possibilities, e.g.:

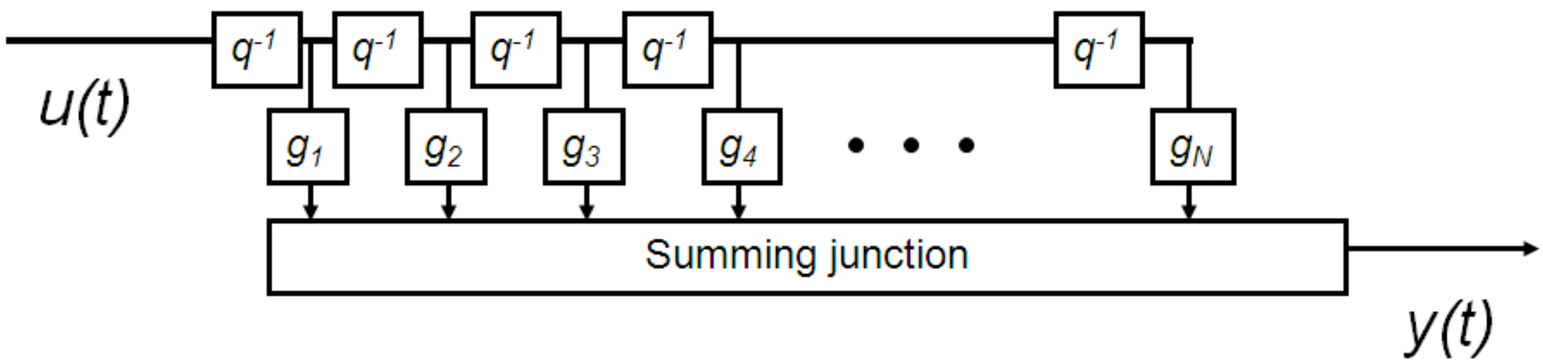
1. Finite Impulse Response (FIR model)
2. Laguerre expansion
3. Auto-Regressive Moving Average (ARMAX)
4. State Space System

1. Finite Impulse Response (FIR)

$$y(t) = \underbrace{\sum_{k=1}^{N} g(k)u(t-k)}_{FIR} + \underbrace{\sum_{k=N+1}^{\infty} g(k)u(t-k)}_{Mismatch = \Delta(q^{-1})u(t)}$$



Typically: *N=30-50*



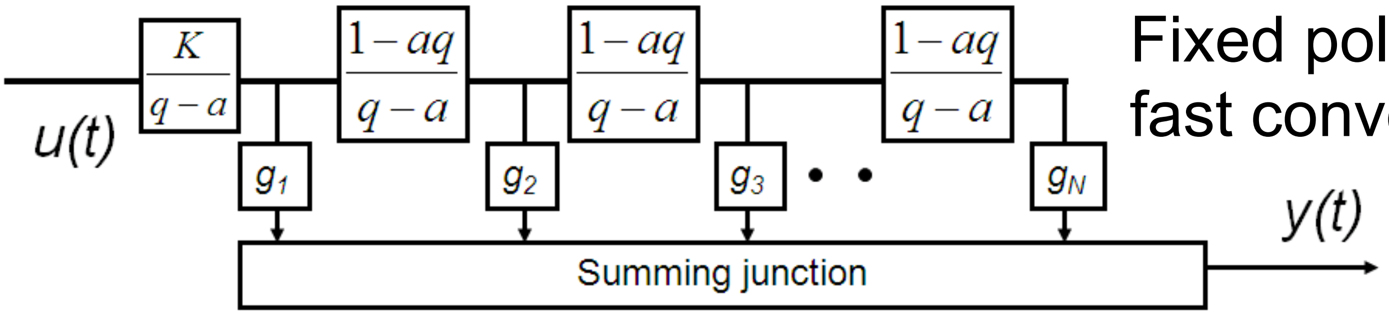
$$y(t) = \varphi(t-1)^T \theta + v(t)$$
$$\varphi(t-1)^T = \left(u(t-1),\dots,u(t-N)\right)$$
$$\theta^T = (g_1,\dots,g_N)$$

Used for MPC identification by

- Aspentech
- Emerson Ovation/Delta V
- Exxon

2. Laguerre Model

$$y(t) = \sum_{k=0}^{N} g(k) \frac{K}{q-a} \left(\frac{1-aq}{q-a} \right)^k u(t) \qquad 0 \le a < 1$$



Fixed pole chosen to give fast convergence

$$y(t) = \varphi(t-1)^T \theta + v(t)$$

$$\varphi(t-1)^T = \left(L_0 u(t-1), \ldots, L_N u(t-N) \right), \qquad L_i(q^{-1}) - \text{Laguerre filters}$$

$$\theta^T = (g_1, \ldots, g_N)$$

Universal Dynamics/ILS

3. ARMAX Model

$$y(t) = G(q^{-1})u(t) + H(q^{-1})v(t) \qquad G(q) = \frac{B(q)}{A(q)} \quad H(q) = \frac{C(q)}{D(q)}$$

$$D(q^{-1})A(q^{-1})y(t) = D(q^{-1})B(q^{-1})u(t) + A(q^{-1})C(q^{-1})v(t)$$

$$\boxed{A'(q^{-1})y(t) = B'(q^{-1})u(t) + C'(q^{-1})v(t)}$$

$$y(t) = \varphi(t-1)^T \theta + v(t)$$

$$\varphi(t-1)^T = \big(y(t-1),\ldots, y(t-n), u(t-1),\ldots, u(t-m), v(t-1),\ldots, v(t-l)\big)$$

$$\theta^T = (a_1,\ldots,a_n, b_1,\ldots,b_m, c_1,\ldots,c_l)$$

Mostly for Academic Research

Relationship Between ARMAX and State Space Models

$$x(t+1) = Ax(t) + Bu(t)$$
$$y = Cx(t) + Du(t)$$

⟺

$$G(q) = C\left(qI - A\right)^{-1} B + D$$

Many choices of A,B,C,D are possible

1. Controller form
2. Observer form
3. Balanced realization
4. Diagonal form
5. Derived from material and energy balances by linearization

Related through non-singular (similarity) transformations

$$\Gamma x(t+1) = (\Gamma A \Gamma^{-1})\Gamma x(t) + (\Gamma B)u(t)$$
$$y = (C\Gamma^{-1})\Gamma x(t) + Du(t)$$

New state $z(t) = \Gamma x(t)$

Reachability/Observability

$$\text{rank}\left[B \mid AB \mid A^2B \mid \cdot \mid A^{n-1}B\right] = n$$

Reachability: Any state can be reached in a finite amount of time

$$y(t) = \frac{B(q)}{A(q)}u(t)$$

Observability: Any state can be determined in a finite amount of time

$$\text{rank}\begin{bmatrix} C \\ CA \\ CA^2 \\ \cdot \\ CA^{n-1} \end{bmatrix} = n$$

- *A(q)* and *B(q)* no common factors = Observable+Controllable (Co-prime)
- *A(q)* and *B(q)* no common unstable factors = Detectable+Stabilizable

Detectable: Any unstable state is observable
Stabilizable: Any unstable state is reachable

Very Simple Examples

Problem 1: Determine (identify) *b* from data. $y(t) = bu(t-1)$

$$\hat{b} = \frac{y(t)}{u(t-1)}$$

For identification we need excitation, i.e.

$$u(t-1) \neq 0$$

Problem 2: **I**dentify *a* from data. $y(t) = ay(t-1) + u(t-1)$

$$\hat{a} = \frac{y(t) - u(t-1)}{y(t-1)}$$

For identification we need $y(t-1) \neq 0$

More Complex Example

Problem 3: Determine *a* and *b* from data.

$$y(t) = ay(t-1) + bu(t-1)$$

$$y(t) = \phi(t-1)^T \theta$$

$$\phi(t-1) = \left(y(t-1), u(t-1)\right)^T$$

$$\theta = (a, b)^T$$

$$y(t) = \varphi(t-1)^T \theta$$

$$\underbrace{\begin{pmatrix} y(1) \\ y(2) \end{pmatrix}}_{Y} = \underbrace{\begin{pmatrix} \varphi(0) \\ \varphi(1) \end{pmatrix}^T}_{\Phi^T} \theta \quad \Longrightarrow \quad Y = \Phi^T \theta$$

$$\theta = \Phi^{-T} Y \qquad \det \Phi \neq 0$$

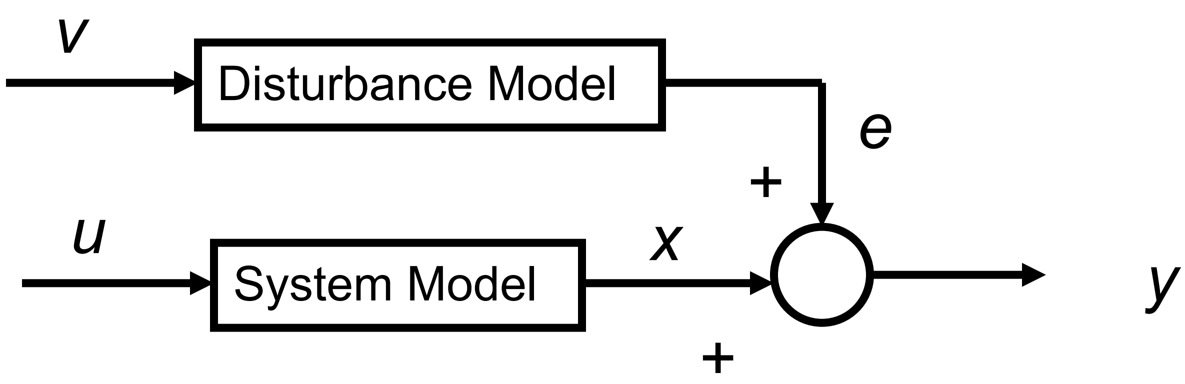
For identification we need (1) $\det \Phi \neq 0$ Excitation

(2) $\{A, B\}$ Co prime

For more than
2 data points
Solve least squares

$$\theta = \left(\Phi \Phi^T\right)^{-1} \Phi Y$$

Least Squares Identification with Disturbances



$$y(t) = \underbrace{\frac{B(q^{-1})}{A(q^{-1})}u(t)}_{\text{system response } x(t)} + \underbrace{\frac{C(q^{-1})}{D(q^{-1})}v(t)}_{\text{disturbance response } e(t)}$$

Problem: Find $\{A, B, C, D\}$ from the data.

Many Methods use Batch estimation

1. Non-convex optimization (real problem)
2. Equation error approach
3. Output error (local NLP)
4. Instrumental variables
5. Approximate maximum likelihood
6. Sub-space identification
7. Many more,…

Approximarions

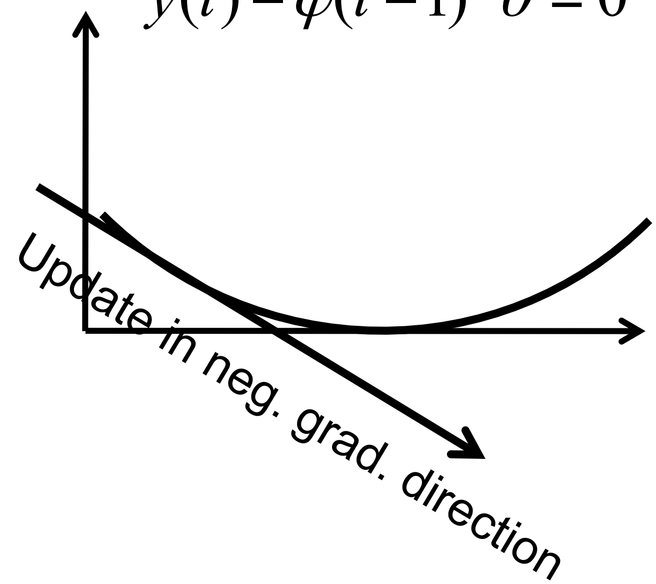
FIR/Step-response and Laguerre methods are solved to global optimality using a linear method like least squares and excitation is easier (step,impulse, prbs).

System Identification for the User by Lennart Ljung

Recursive Identification: The Gradient Method

Convert to regression model

$$A(q^{-1})y(t) - B(q^{-1})u(t-d) = 0 \quad \Rightarrow \quad y(t) - \varphi(t-1)^T\theta = 0$$



$$\min_{\hat{\theta}} J(\hat{\theta}), \qquad J(\hat{\theta}) = (y(t) - \varphi(t-1)^T\hat{\theta})^2$$

$$\frac{\partial J(\hat{\theta})}{\partial \hat{\theta}} = -\varphi(t-1)\underbrace{(y(t) - \varphi(t-1)^T\hat{\theta})}_{\text{error} = e(t)}$$

$$\boxed{\hat{\theta}(t) = \hat{\theta}(t-1) + \mu(t)\ \varphi(t-1)e(t) \quad \text{gradient update}}$$

This is the first attempt. Notice that it is not stable unless we adapt $\mu(t)$

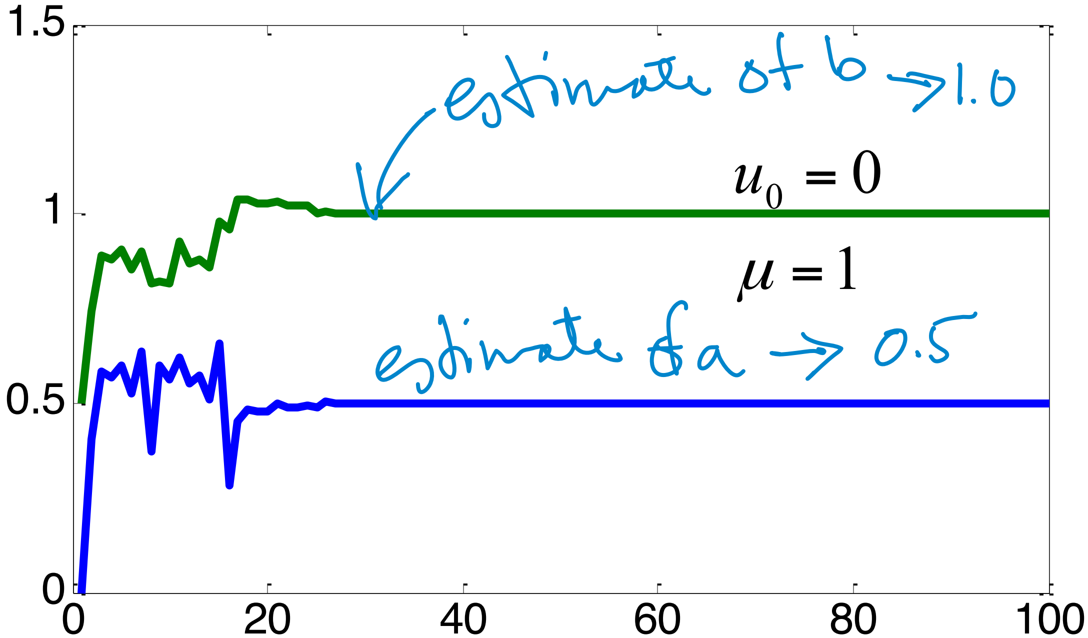
Simulation– Gradient Algorithm

$$y(t) = 0.5y(t-1) + u(t-1)$$

True a=0.5, true b= 1.

$$u = u_0 + v(t)$$

v(t) = RAND Uniformly distributed pseudo-random numbers (Matlab).



Unstable if *u* and/or y are too large!

NOT SUITABLE FOR ADAPTIVE CONTROL

unless we adapt gain as shown next

Matlab program Gradient

Stability of Gradient Algorithm

$$V(t)=\frac{1}{2}\tilde{\theta}(t)^T\tilde{\theta}(t),\qquad \tilde{\theta}(t)=\theta-\hat{\theta}(t)\quad \text{Lyapunov function}$$

$$\tilde{\theta}(t)=\tilde{\theta}(t-1)-\mu(t)\ \varphi(t-1)e(t)$$

This algorithm is used for adaptive control

$$V(t)=V(t-1)-\mu\left(1-\tfrac{1}{2}\mu\varphi(t-1)^T\varphi(t-1)\right)e(t)^2$$

$$0<\mu<\frac{2}{\varphi(t-1)^T\varphi(t-1)}$$

$$\hat{\theta}(t)=\hat{\theta}(t-1)+\underbrace{\left(\frac{p}{c+\varphi(t-1)^T\varphi(t-1)}\right)}_{\mu\,=\,\textbf{Adaptive Gain}}\varphi(t-1)e(t)\qquad \text{Kaczmarz algorithm}$$

$0<c,\qquad 0<p<2$ ← tuning parameters

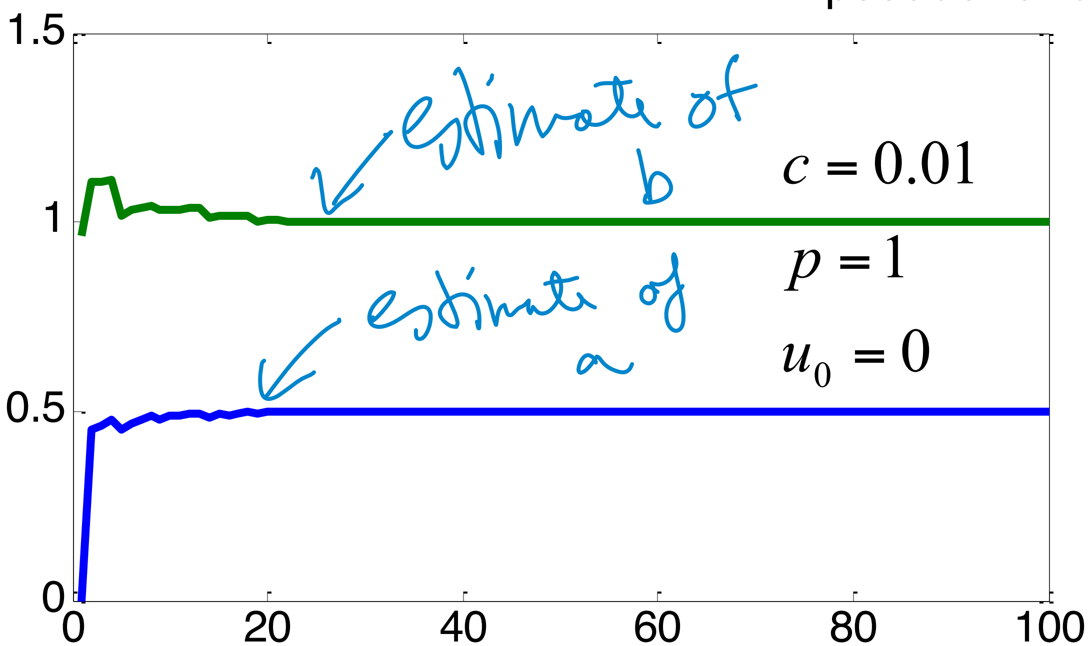
Matlab program kaczmarz

Simulation—Gradient Algorithm

$$y(t) = 0.5y(t-1) + u(t-1)$$

$$u = 1 + v(t)$$

v(t) = RAND   Uniformly distributed pseudo-random numbers (Matlab).



$c = 0.01$

$p = 1$

$u_0 = 0$

Stable for any magnitude *u* and y!

SUITABLE FOR ADAPTIVE CONTROL?

Yes

Matlab program kaczmarz

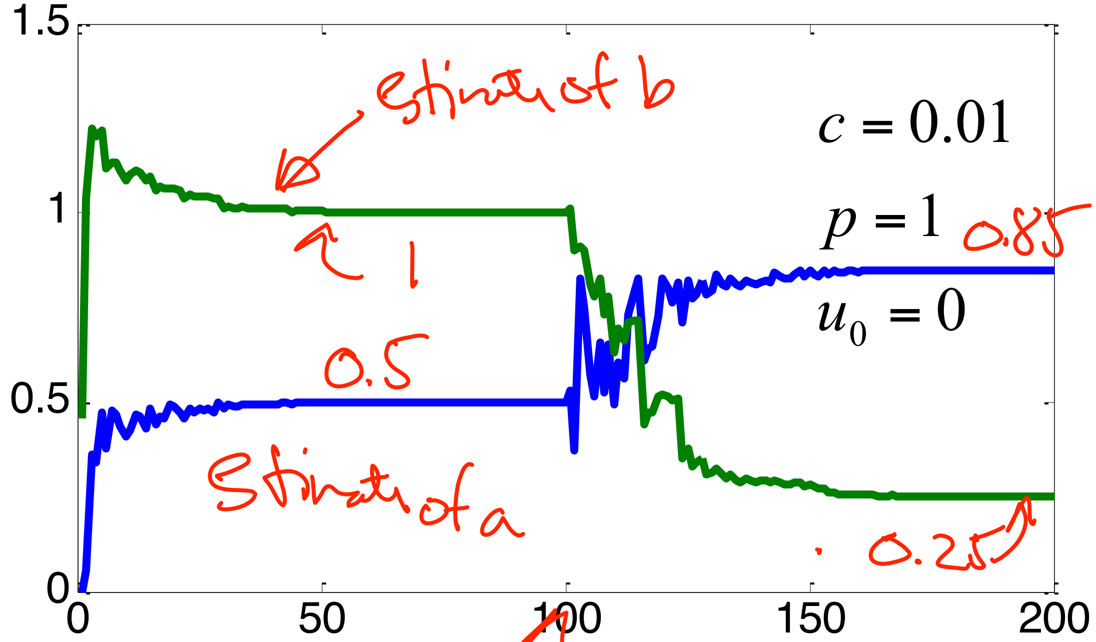
Simulation– Gradient Algorithm

$$y(t) = 0.5y(t-1) + u(t-1), \qquad t \le 100$$

$$y(t) = 0.85y(t-1) + 0.25u(t-1), \qquad t > 100$$

$$u = u_0 + v(t)$$

model changes when setpoint changes to simulate nonlinear behavior



- Stable
- Easy to implement
- Recursive
- Adaptive (tracks parameters)

Matlab program kaczmarz

1-Step Ahead Control
(Special case of Minimum Variance Control)

$$y(t) = \frac{B(q^{-1})}{A(q^{-1})} u(t-d)$$

Problem: Choose control so that *y(t+1)=y(t+1)**

$$A(q^{-1}) = 1 - aq^{-1}, \qquad B(q^{-1}) = b, \qquad d = 1$$

$$y(t) = ay(t-1) + bu(t-1)$$

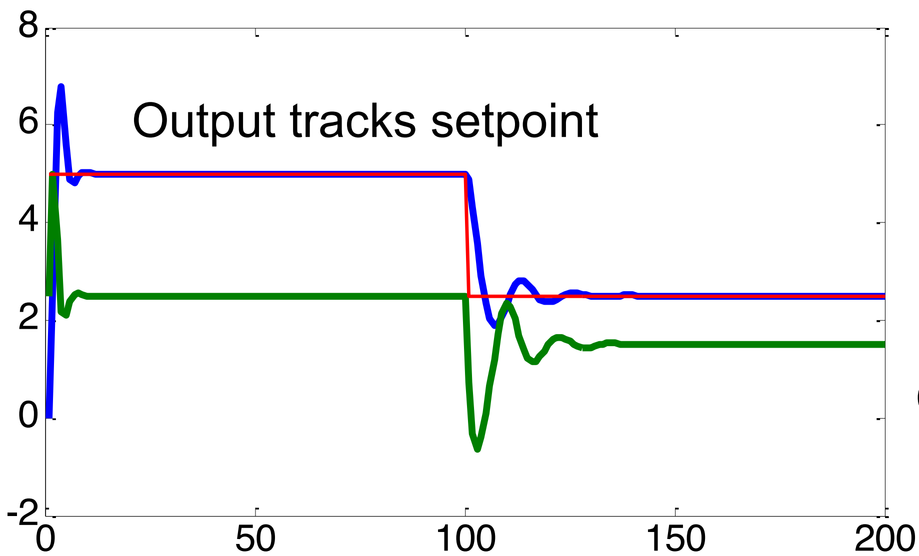
Solve for $u: y(t+1)^* = ay(t) + bu(t)$

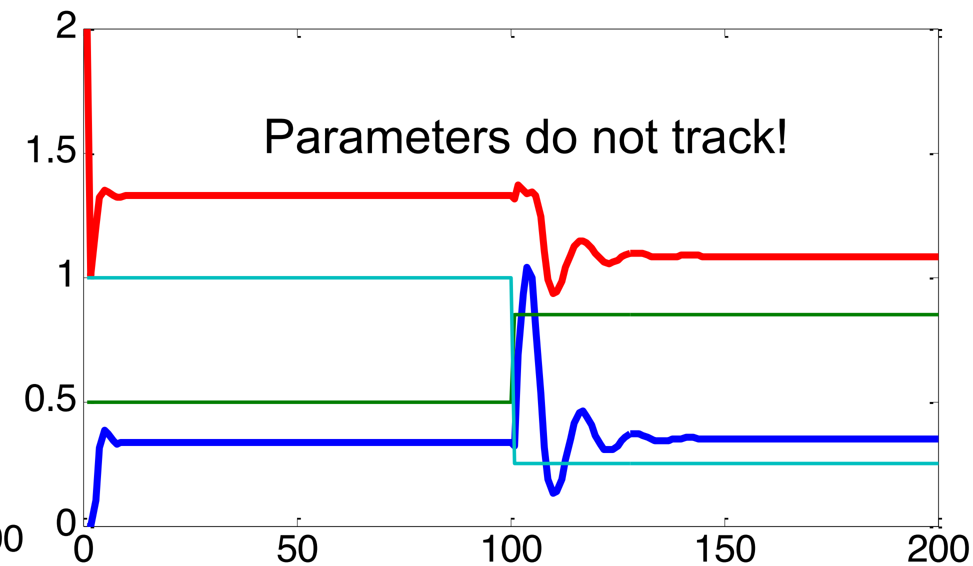
$$u(t) = \frac{1}{b}\left(y(t+1)^* - ay(t)\right), \quad \text{need} \quad b \neq 0$$

Adaptive Case

- Estimate a and b using gradient algorithm
- Project estimate so that $0 < b_{\min} \leq \hat{b}, \qquad b_{\min} \leq b$
- Implement 1-step ahead control

$$u(t) = \frac{1}{\hat{b}}\left(y(t+t)^* - \hat{a}y(t)\right)$$





Not enough excitation to estimate parameters

1-Step ahead -Second Example

Process $$y(t) = ay(t-1) + b_1 u(t-1) + b_2 u(t-2)$$

Control $$u(t) = \frac{1}{b_1}\left(y(t+1)^* - ay(t) - b_2 u(t-1)\right)$$

Closed loop

$$y(t+1) = y(t+1)^*$$

$$u(t) = \frac{1}{b_1}\left(y(t+1)^* - ay(t)^*\right) - \frac{b_2}{b_1} u(t-1)$$

$$\text{Must have:} \quad \left|\frac{b_2}{b_1}\right| < 1 \quad \text{to get stable controller!}$$

In general must have 1/B stable

---plant must be stably invertible (minimum phase)

Recursive Least Squares

minimize : $J(t) = \sum_{1}^{t} \left(y(i) - \phi(i-1)^T \theta(t) \right)^2 r^{-1} + \left(\theta(t) + \theta(0) \right)^T P(0)^{-1} \left(\theta(t) + \theta(0) \right)$

Recursive Solution :

$$e(t) = y(t) - \phi(i-1)^T \theta(t-1)$$

$$P(t) = \frac{1}{\lambda} \left(P(t-1) - P(t-1)\phi(i-1)\phi(i-1)^T P(t-1) / (r + \phi(i-1)^T P(t-1)\phi(i-1)) \right)$$

$$\theta(t) = \theta(t-1) + P(t)\phi(i-1)e(t)$$

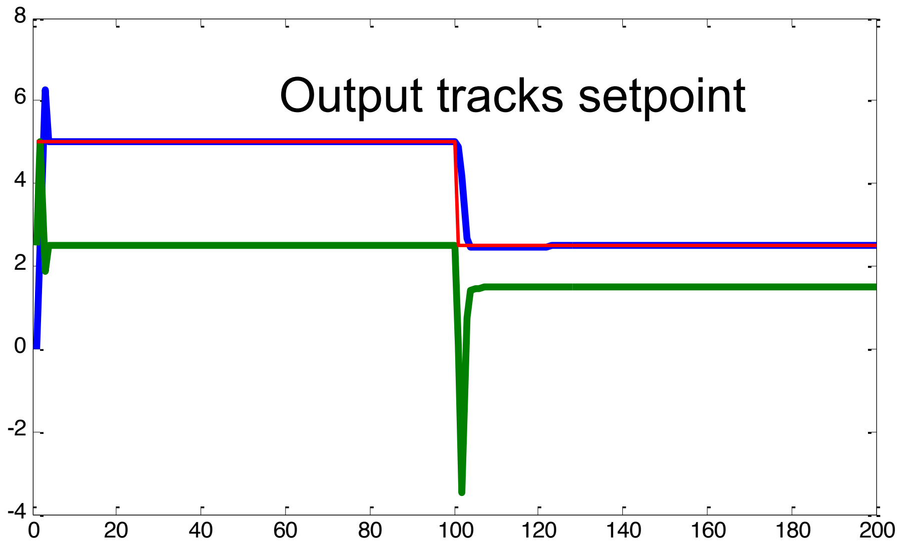
Important Modifications:

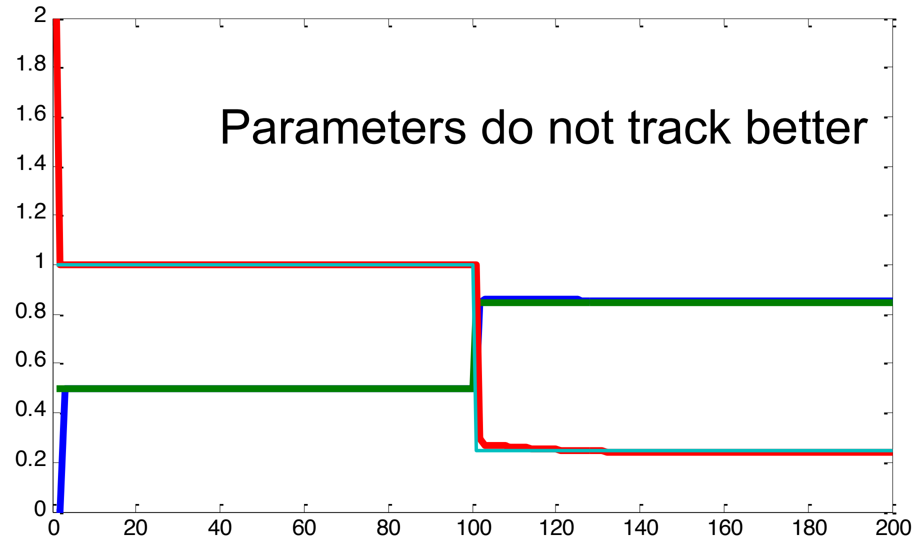
- Covariance Reset
- Variable Forgetting factor
- Adaptive Deadzone
- UDU factorization for numerical stability

Adaptive Case (same as for gradient)

- Estimate a and b using rls algorithm
- Project estimate so that $0 < b \leq \hat{b}$
- Implement 1-step ahead control

$$u(t) = \frac{1}{\hat{b}}\left(y(t+t)^* - \hat{a}y(t)\right)$$





Not enough excitation to estimate parameters

Adaptive Control Stability

$$e(t) = y(t) - \hat{b}(t-1)u(t-1),$$ error

$$\hat{b}(t) = \hat{b}(t-1) + \underbrace{\left(\frac{p}{c + u(t-1)^2}\right)}_{\mu = \text{Adaptive Gain}} u(t-1)e(t)$$ gradient estimate

$$u(t) = \frac{y^*(t+1)}{\hat{b}(t)}$$

$$V(t) = \left(b(t) - \hat{b}(t)\right)^2$$ Lyapunov Fn

Stability:
One step Adaptive Control with Gradient Algorithm

$$e(t) = y(t) - \phi(t-1)^T \hat{\theta}(t-1)$$

$$\hat{\theta}(t) = \hat{\theta}(t-1) + \frac{p\phi(t-1)}{c + \phi(t-1)^T \phi(t-1)} e(t)$$

$$0 < p \leq 2, \qquad 0 \leq c < \infty$$

$$\text{Solve for} \quad u(t) : y(t+1)^* = \phi(t)^T \hat{\theta}(t)$$

Theorem (stably invertible plant, admissible model):

$$\lim_{t \to \infty} (y(t) - y(t)^*) = 0, \quad |u(t)| \leq K$$

Proof : Use $V(t) = (\theta - \hat{\theta}(t))^T (\theta - \hat{\theta}(t))$ as Lyapunov function

(Goodwin et al, 1980)

Stability:
One step Adaptive Control with RLS

$$\textbf{minimize :} \quad J(t) = \sum_{1}^{t} \left(y(i) - \phi(i-1)^T \theta(t) \right) r^{-1} + \left(\theta(t) + \theta(0) \right)^T P(0)^{-1} \left(\theta(t) + \theta(0) \right)$$

Recursive Solution :

$$e(t) = y(t) - \phi(i-1)^T \theta(t-1)$$

$$P(t) = \frac{1}{\lambda} \left(P(t-1) - P(t-1)\phi(i-1)\phi(i-1)^T P(t-1) / (r + \phi(i-1)^T P(t-1)\phi(i-1)) \right)$$

$$\theta(t) = \theta(t-1) + P(t)\phi(i-1)e(t)$$

$$\textbf{Solve for} \quad u(t) : y(t+1)^* = \phi(t)^T \hat{\theta}(t)$$

Theorem (Goodwin et al.1980, stably invertible plant, admissible model):

$$\lim_{t \to \infty} (y(t) - y(t)^*) = 0, \qquad | u(t) | \leq K$$

Proof : Use $V(t) = (\theta - \hat{\theta}(t))^T P(t)^{-1} (\theta - \hat{\theta}(t))$ as Lyapunov function